Chapter 15
Quantum Coherence and Measurement Theory

Abstract The feature of quantum mechanics which most distinguishes it from classical mechanics is the coherent superposition of distinct physical states. Many of the less intuitive aspects of the quantum theory can be traced to this feature. Does the superposition principle operate on macroscopic scales? The famous Schroedinger cat argument highlights problems of interpretation were macroscopic superposition states allowed. In this chapter we discuss schemes to produce and detect superposition states in an optical context. We shall show that such states are very fragile in the presence of dissipation and rapidly collapse to a classical mixture exhibiting no unusual interference features.

The superposition principle is also the source of the "problem of measurement" in quantum mechanics. We do not attempt to present a solution to this problem here. Rather we show how the effect of the environment on superposition states enables a consistent description of the measurement process to be given and which avoids some of the problems inherent in previous approaches.

15.1 Quantum Coherence

The well known two slit experiment demonstrates the observational consequences of the coherent superposition of states, namely the possibility of interference patterns. In analogy with the classical theory of wave interference the visibility of the interference pattern in the probability distributions for various measurements can be used as a measure of quantum coherence. (There is a possibility of confusion here which should be cleared up. In the interference of waves we are usually concerned with the interference of two or more field modes. In this chapter however we are concerned with the superposition of different states of a single mode.)

The essential point in understanding quantum coherence is the physical distinction between the coherent superposition state

$$|\psi> = \sum_j c_j|\phi_j> \tag{15.1}$$

and the classical mixture

$$\rho_m = \sum_j |c_j|^2 \, |\phi_j >< \phi_j| \quad . \tag{15.2}$$

The density operator corresponding to the pure state in (15.1) is

$$\rho_p = \rho_m + \sum_{i \neq j} c_i c_j^* |\phi_i >< \phi_j| \tag{15.3}$$

How is one to distinguish these states in practice? Let X be the operator corresponding to some physical quantity with eigenvalues x. The probability distribution for X in the state $|\psi>$ is then given by

$$P_p(x) = P_m(x) + \sum_{i \neq j} c_i c_j^* < x|\phi_i >< \phi_j|x > , \tag{15.4}$$

where $P_m(x)$ is the probability distribution for the state ρ_m given by

$$P_m(x) = \sum_i |c_i|^2 | < x|\phi_i > |^2 = \sum_i |c_i|^2 P_{\phi_i}(x) \tag{15.5}$$

Measurements of X will distinguish the states ρ_m and ρ_p provided the second term in (15.3) is nonzero. We are thus led to define the quantum coherence with respect to the measurement of X by the coherence function

$$\mathscr{C}(x) = \sum_{i \neq j} c_i c_j^* < x|\phi_i >< \phi_j|x > \quad . \tag{15.6}$$

How does one choose an operator X such that the resulting probability distribution will exhibit interference fringes? Clearly one cannot choose operators which are diagonal in the basis $\{|\phi_i >\}$ as then the coherence function vanishes. The simplest example of an operator which distinguishes these states is the projector

$$P = |\psi >< \psi| , \tag{15.7}$$

with eigenvalues $p \in \{0, 1\}$. Then

$$P_p(p) = \delta_{1,p} \tag{15.8}$$

while

$$P_m(p) = \begin{cases} \sum_j |c_j|^4 & \text{if} p = 1 \\ 1 - \sum_j |c_j|^4 & \text{if} p = 0 \end{cases} \tag{15.9}$$

The coherence function is

$$\mathscr{C}(p) = (-1)^p (\sum_j |c_j|^4 - 1) \tag{15.10}$$

In practice however there may be no way to measure the operator P.

In quantum optics one either measures photon number (by photon counting) or quadrature phase (by balanced homodyne detection). As an example consider the superposition of two number states

$$|\psi>=\frac{1}{\sqrt{2}}(|n_1>+|n_2>)\,, \tag{15.11}$$

and the corresponding mixed state

$$\rho_m=\frac{1}{2}(|n_1><n_1|+|n_2><n_2|)\ . \tag{15.12}$$

A measurement of photon number will not distinguish these states, however as the quadrature phase does not commute with the number operator, quadrature phase measurements should distinguish the states.

Define the quadrature operator

$$X_\theta=(a\mathrm{e}^{-\mathrm{i}\theta}+a^\dagger\,\mathrm{e}^{\mathrm{i}\theta}) \tag{15.13}$$

with eigenstates $|x_\theta>$. Using the result

$$<x_\theta|n>=(2\pi)^{-1/4}(2^n n!)^{-1/2}H_n(\frac{x_\theta}{\sqrt{2}})e^{-x_\theta^2/4-in\theta} \tag{15.14}$$

one finds

$$P_m(x_\theta)=\frac{1}{2}(P^{(1)}(x_\theta)+P^{(2)}(x_\theta)) \tag{15.15}$$

and

$$\mathscr{C}(x_\theta)=(2\pi)^{-1/2}\mathrm{e}^{-x_\theta^2/2}(2^{n_1+n_2}n_1!n_2!)^{-1/2} \tag{15.16}$$

$$\times H_{n_1}(\frac{x_\theta}{\sqrt{2}})H_{n_2}(\frac{x_\theta}{\sqrt{2}})\cos((n_1-n_2)\theta) \tag{15.17}$$

where

$$P^{(i)}(x_\theta)=(2\pi)^{-1/2}(2^{n_i}n_i!)^{-1}H_{n_i}(\frac{x_\theta}{\sqrt{2}})^2\mathrm{e}^{-x_\theta^2/2} \tag{15.18}$$

Thus a superposition of number states will exhibit interference fringes for some quadrature phase angle θ. In Fig. 15.1 we plot $P_p(x_\theta)$ versus x_θ for $\theta=0$. It is surprising that, depending on whether n_1-n_2 is even or odd, certain phase angles do not give interference. However it is quite clear that the superposition of two number states will exhibit phase dependent noise despite the fact that number states themselves have phase independent noise.

As a second example consider the superposition of two coherent states (so called *cat states*)

$$|\psi>=\mathscr{N}(|\alpha_1>+|\alpha_2>)\,, \tag{15.19}$$

where

$$\mathscr{N}=\left(2+\mathrm{e}^{-\frac{1}{2}(|\alpha_1|^2+|\alpha_2|^2)}(\mathrm{e}^{\alpha_1^*\alpha_2}+\mathrm{e}^{\alpha_1\alpha_2^*})\right)^{-1/2} \tag{15.20}$$

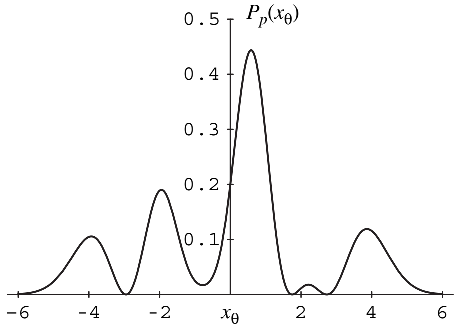

Fig. 15.1 A plot of the probability density for the quadrature phase with phase angle at zero for a superposition of two number states, with $n_1 = 0, n_2 = 5$

and

$$\rho_m = \frac{1}{2}(|\alpha_1 >< \alpha_1| + |\alpha_2 >< \alpha_2|) \tag{15.21}$$

In Chap. 17 we discuss how states of this kind have been produced for the centre of mass degree of freedom for a harmonically trapped ion. For cat states, measurements of either quadrature phase or photon number should exhibit interference fringes as neither of the corresponding operators commutes with a coherent state projector. However there is an optimal quadrature phase angle which leads to maximum interference.

In the case of photon number one finds

$$P_p(n) = \mathcal{N}^2(P^{(1)}(n) + P^{(2)}(n) + \mathscr{C}(n)) \tag{15.22}$$

where

$$\mathscr{C}(n) = (\frac{1}{n!})\exp\left(\frac{-1}{2}(|\alpha_1|^2 + |\alpha_2|^2)\right)((\alpha_1\alpha_2^*)^n + (\alpha_1^*\alpha_2)^n) \tag{15.23}$$

where

$$P^{(i)}(n) = (\frac{1}{n!})|\alpha_i|^{2n}e^{-|\alpha_i|^2} \tag{15.24}$$

If we write $\alpha_i = |\alpha_i|e^{i\phi}$ then

$$\mathscr{C}(n) = 2(P^{(1)}(n)P^{(2)}(n))^{1/2}\cos(n(\phi_1 - \phi_2)) \tag{15.25}$$

and we see that the degree of interference depends on the phase angle between the amplitudes of the superposed states. For simplicity let us take $|\alpha_1| = |\alpha_2| = |\alpha|$. Then

$$P_p(n) = 2\mathcal{N}^2(\frac{1}{n!})|\alpha|^{2n}e^{-|\alpha|^2}(1 + \cos(n(\phi_1 - \phi_2)) . \tag{15.26}$$

When $\phi_1 - \phi_2 = \pi$, $P_p(n)$ is zero for n odd. Thus a superposition of coherent states π out of phase but of equal amplitude will contain only even photon number, a similar situation to that of a squeezed vacuum state.

In the case of quadrature phase measurements one finds

$$P^{(i)}(x_\theta) = (2\pi)^{-1/2}\exp\left(-|\alpha_i|^2 + \frac{x_\theta}{2} - \frac{(x_\theta - \alpha_i e^{-i\theta})^2}{2} - \frac{(x_\theta - \alpha_i^* e^{i\theta})^2}{2}\right) \quad (15.27)$$

and thus

$$\mathscr{C}(x_\theta) = (\frac{2}{\pi})^{1/2}\Re\exp\Big(\frac{-1}{2}(|\alpha_1|^2 + |\alpha_2|^2 - x_\theta^2 + (x_\theta - \alpha_1 e^{-i\theta})^2 + (x_\theta - \alpha_2^* e^{i\theta})^2)\Big) \quad (15.28)$$

To gain some insight into these equations we take $|\alpha_1| = |\alpha_2| = |\alpha|$ and choose

$$\theta = \phi_+ = (\phi_1 + \phi_2)/2 \quad (15.29)$$

This phase angle bisects the angle between the two coherent states (see Fig. 15.2). We then have

$$P^{(1)}(x) = P^{(2)}(x) = (2\pi)^{-1/2}\exp\left(-(x - 2|\alpha|\cos\phi_-)^2/2\right) \quad (15.30)$$

and

$$\mathscr{C}(x) = 2P^{(1)}(x)\cos(2|\alpha|\sin\phi_-(x - |\alpha|\cos\phi_-)) \quad (15.31)$$

with $\phi_- = (\phi_1 - \phi_2)/2$ and we have put $x = x_{\phi_+}$. Thus

$$P_p(x) = 2\mathscr{N}^2 P^{(i)}(x)(1 + \cos(2|\alpha|\sin\phi_-(x - |\alpha|\cos\phi_-)) \quad (15.32)$$

This is a Gaussian centred at $2|\alpha|\cos\phi_-$ modulated by an interference envelope (see Fig. 15.3)

This result has a simple geometric interpretation. Referring to Fig. 15.2 we see that projecting the two coherent states onto the x_{ϕ_+} axis gives a maximum overlap centred on the mean value $< X_{\phi_+} >= 2|\alpha|\cos\phi_-$. We conclude that whenever the coherent states are projected onto a quadrature such that they overlap exactly, the interference will be maximal. Conversely we expect that if we project the coherent

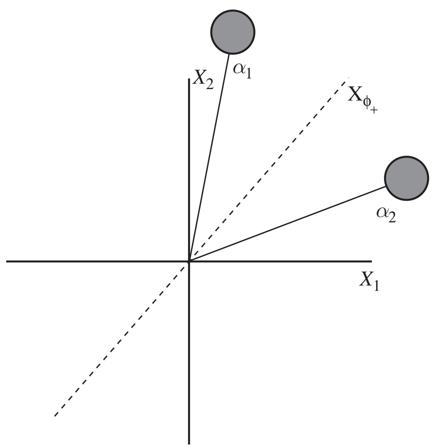

Fig. 15.2 Phase space representation of the superposition of two coherent states. The dashed line represents the direction of the quadrature phase angle which exhibits maximum interference

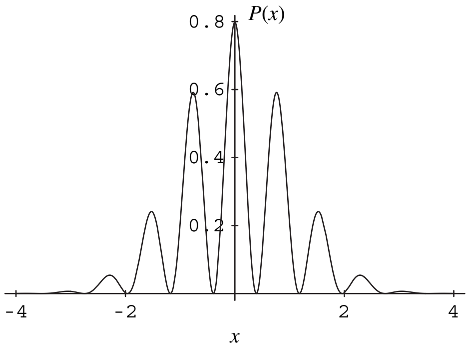

Fig. 15.3 The probability density for the quadrature phase at phase angle $\theta = \phi_+ = \pi/2$ for a superposition of two coherent states with $\phi_- = \pi/2$. This is the quadrature direction which bisects the angle between the two superposed coherent amplitudes, and which exhibits maximum interference

states onto a quadrature with $\theta = \phi_+ \pm \pi/2$ there will be minimum overlap and thus the least interference, (see Exercise 15.1).

We now need to consider the effect of dissipation on the interference features discussed above. With this in mind we write the coherence function in terms of the complex valued functions $C(x)$,

$$\mathscr{C}(x) = C(x) + C(x)^* \tag{15.33}$$

where for a general superposition state

$$\mathscr{C}(x) = \sum_{i<j} c_i c_j^* < x|\phi_i >< \phi_j|x > \tag{15.34}$$

A convenient measure of the degree of quantum coherence is the quantum visibility defined by

$$\mathscr{V}(x) = \frac{|C(x)|}{(P^{(1)}(x)P^{(2)}(x))^{1/2}} \tag{15.35}$$

In all the cases considered above $\mathscr{V}(x)$ is unity for all values of x and the states considered thus have maximum quantum coherence. However when dissipation is present this is no longer the case.

15.2 The Effect of Dissipation

In this section we show that quantum coherence associated with superposition states is extremely fragile in the presence of nonunitary effects such as damping. Such effects cause a decay of quantum coherence at a rate which is proportional to a parameter which measures the separation of the superposed states. For macroscopic separations this decay can be very rapid.

We first consider the effects of dissipation. The master equation for a damped harmonic oscillator in the interaction picture is

$$\frac{d\rho}{dt} = \frac{\gamma}{2}(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) \tag{15.36}$$

where γ is the damping constant and the reservoir is taken to be at zero temperature. We will solve this equation for an initial superposition of coherent states using the normally ordered characteristic function,

$$X(\lambda, t) = tr(\rho(t) e^{\lambda a^\dagger} e^{-\lambda^* a}) \tag{15.37}$$

Using (15.36) the equation of motion for $X(\lambda, t)$ is

$$\frac{\partial X}{\partial t} = -\frac{\gamma}{2}(\lambda^* \frac{\partial X}{\partial \lambda} + \text{c.c.}) \tag{15.38}$$

where c.c. means complex conjugate. The solution to this equation is

$$X_0(\lambda e^{-\gamma t/2}, \lambda^* e^{-\gamma t/2}) \tag{15.39}$$

where

$$X_0(\lambda, \lambda^*) = X(\lambda, 0) \tag{15.40}$$

For the initial state given in (15.19), a superposition of coherent states, we find

$$X_0(\lambda, \lambda^*) = \mathscr{N}^2 \sum_{i,j=1}^{2} e^{(\lambda \alpha_i^* - \text{c.c.})} < \alpha_i | \alpha_j > \tag{15.41}$$

Thus

$$X(\lambda, t) = \mathscr{N}^2 \sum_{i,j=1}^{2} < \alpha_i | \alpha_j > \exp\left((\lambda \alpha_i^* - \text{c.c.}) e^{-\gamma t/2}\right) \tag{15.42}$$

The corresponding solution for the density operator is

$$\rho(t) = \mathscr{N}^2 \sum_{i,j=1}^{2} < \alpha_i | \alpha_j >^{(1-e^{-\gamma t})} | \alpha_i e^{-\gamma t/2} > < \alpha_j e^{-\gamma t/2} | \tag{15.43}$$

Due to the coefficient with $i \neq j$ in this expansion the contribution of the off-diagonal terms to the coherence function will be small. Thus the superposition is reduced to a near mixture of coherent states.

The visibility for any measurement is easily found to be

$$\mathscr{V}(x) = | < \alpha_1 | \alpha_2 > |^{(1-e^{-\gamma t})} \tag{15.44}$$

where $x = x_\theta$ for quadrature phase or $x = n$ for number measurements. For short times this is given approximately by

$$\mathscr{V}(x) \approx \exp(-\frac{\gamma t}{2} |\alpha_1 - \alpha_2|^2) \tag{15.45}$$

Thus the rate at which coherence decays is proportional to the square of the distance between the superposed coherent amplitudes. In the last section we considered the case $\alpha_1 = -\alpha_2$ which gave good interference fringes centered on the origin for the appropriate quadrature. However we now see that such fringes must decay rapidly causing the quadrature phase statistics to be indistinguishable from a classical mixture of coherent states.

This result is not confined to amplitude damping. For example the phase diffusion model discussed in Exercise 6.3 for which the master equation is

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\gamma}{2}(2a^\dagger a\rho a^\dagger a - (a^\dagger a)^2\rho - \rho(a^\dagger a)^2) \tag{15.46}$$

The solution for the matrix elements in the number basis is

$$< n|\rho(t)|m >= \exp(-\frac{\gamma}{2}t(n-m)^2) < n|\rho(0)|m > \tag{15.47}$$

If $\rho(0)$ is a superposition of number states $|n_1>$ and $|n_2>$ the visibility of the quadrature phase statistics is given by

$$\mathscr{V}(x) = \exp(-\frac{\gamma t}{2}(n_1 - n_2)^2) \tag{15.48}$$

which shows a rapid decay when $n_1 - n_2$ is large.

These examples are special; the visibility in each case decays in the same way for all measurements which give interference fringes. This is because the superposed states are eigenstates or near eigenstates of the operator appearing in the irreversible part of the evolution equation. This is an important point to which we shall return in the next section. In other cases the visibility is more complicated. For example consider the effect of phase diffusion on the quadrature phase statistics of a superposition of two coherent states with $\alpha_1 = -\alpha_2 = q_0$ where q_0 is real. If we choose $\theta = \pi/2$ (that is we project onto the imaginary axis in the complex amplitude diagram) we find that for short times

$$\mathscr{V}(x_{\pi/2}) \approx 1 - \gamma t(2q_0 x_{\pi/2})^2 \tag{15.49}$$

As expected the coherence decays from unity at a rate which is proportional to the square of the separation of the superposed states. In Fig. 15.4 we plot the probability distribution for quadrature phase at $\theta = \pi/2$ with a superposition of coherent states, with $q_0 = 2$, subject to phase diffusion with different damping rates.

A similar result may be derived for quadrature phase measurements on a superposition of number states undergoing damping. To show this we use the complex P-representation for the projector $|n_i >< n_j|$

$$|n_i >< n_j| = \oint_{c_1} d\alpha \oint_{c_2} \mathrm{d}\beta\, P_{ij}(\alpha,\beta)\frac{|\alpha >< \beta^*|}{< \beta^*|\alpha >} \tag{15.50}$$

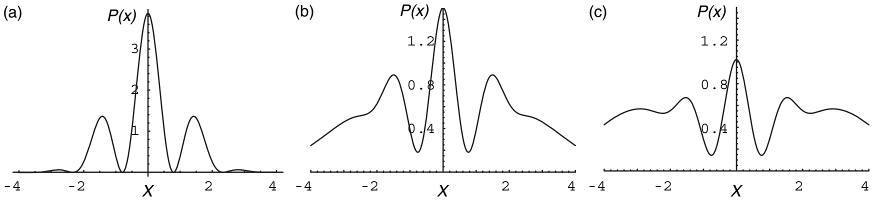

Fig. 15.4 A plot of the probability distribution for quadrature phase, $\theta = \pi/2$, on an oscillator prepared in a coherent superposition of coherent states, $|2\rangle + |-2\rangle$, which is subject to phase diffusion. (**a**) $\gamma t = 0$ (**b**) $\gamma t = 0.1$ (**c**) $\gamma t = 0.5$. The loss of visibility and spreading of the distribution is a conseqence of phase diffusion

where

$$P_{ij}(\alpha,\beta) = -\frac{1}{4\pi^2} e^{\alpha\beta} (n_i! n_j!)^{1/2} \alpha^{-(n_i+1)} \beta^{-(n_j+1)} \tag{15.51}$$

and c_1, c_2 are contours encircling the origin. Thus under time evolution $|n_i><n_j|$ evolves to

$$\begin{aligned}(|n_i><n_j|)_t = &\oint_{c_1} d\alpha \oint_{c_2} d\beta \; P_{ij}(\alpha,\beta) <\beta^*|\alpha>^{(1-e^{-\gamma t})} \\ &\times \frac{|\alpha e^{-\gamma t/2}><\beta^* e^{-\gamma t/2}|}{<\beta^*|\alpha>}\end{aligned} \tag{15.52}$$

Using this result one may show

$$\begin{aligned}C(x_\theta,t) = &(2\pi)^{-1/2} 2^{-(n_1+n_2)/2} (n_1! n_2!)^{-1/2} (-1)^{n_1+n_2} \\ &\exp\left(-\frac{x_\theta^2}{2} - i(n_1-n_2)\theta\right) e^{-\gamma t(n_1+n_2)/2} \\ &\sum_{p=0}^{\min(n_1,n_2)} 2^p e^{\gamma t p} p! \binom{n_1}{p} \binom{n_2}{p} H_{n_1+n_2-2p}\left(\frac{x_\theta}{2}\right)\end{aligned} \tag{15.53}$$

The dominant term in the sum occurs for $p = \min(n_1,n_2)$. For example if $n_1 > n_2$ the time dependence of the dominant term is proportional to $\exp(-\gamma t(n_1-n_2)/2)$. As expected the coherence decays at a rate which is proportional to the separation of the states.

15.2.1 Experimental Observation of Coherence Decay

The Haroche group in Paris demonstrated the rapid decay of coherence for a superposition of two coherent states. They used Rydberg atoms in microwave cavities [1]. Two Rydberg atomic levels with ground state $|g\rangle$ and excited state $|e\rangle$ interact with a cavity field. The cavity field is well detuned from the atomic resonance. The effect of the interaction is to change the phase of the field in the cavity conditional on the atomic state. An effective Hamiltonian for this interaction can be written as

$$H_C = \hbar\chi a^\dagger a\sigma_z \tag{15.54}$$

where $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$.

If the cavity C is initially prepared in a weak coherent state, $|\alpha\rangle$, (in the experiment $|\alpha| = 3.1$) and the atom is prepared in an equal superposition of the ground and excited states, the total system state evolves to

$$|\psi(\tau) = \frac{1}{2}\left(|g\rangle|\alpha e^{i\phi}\rangle + |e\rangle|\alpha e^{-i\phi}\rangle\right) \tag{15.55}$$

where $\phi = \chi\tau$, for an interaction time τ. The state in (15.55) is an entangled state between a two level degree of freedom and an oscillator. To obtain a state which entangles the atomic degree of freedom with coherent superpositions of coherent states we use an independent laser to rotate the atomic states by $|g\rangle \to (|g\rangle + |e\rangle)/\sqrt{2}, \quad |e\rangle \to (|g\rangle - |e\rangle)/\sqrt{2}$. The state at the end of this last pulse is

$$|\psi\rangle_{\text{out}} = \frac{1}{2}\left(|g\rangle(|\alpha \mathrm{e}^{\mathrm{i}\phi}\rangle + |\alpha \mathrm{e}^{-\mathrm{i}\phi}\rangle) + |e\rangle(|\alpha \mathrm{e}^{\mathrm{i}\phi}\rangle - |\alpha \mathrm{e}^{-\mathrm{i}\phi}\rangle)\right) \tag{15.56}$$

If we now measure the atomic state $|g\rangle$ (by state selective ionisation in the experiment) the conditional state of the field is either

$$|\psi^{(g)}\rangle_{\text{out}} = \mathcal{N}_+(|\alpha \mathrm{e}^{\mathrm{i}\phi}\rangle + |\alpha \mathrm{e}^{-\mathrm{i}\phi}\rangle) \tag{15.57}$$

or

$$|\psi^{(e)}\rangle_{\text{out}} = \mathcal{N}_-(|\alpha \mathrm{e}^{\mathrm{i}\phi}\rangle - |\alpha \mathrm{e}^{-\mathrm{i}\phi}\rangle) \tag{15.58}$$

where $\mathcal{N}_\pm$ is the normalisation constant. These conditional states are superpositions of coherent states.

In this discussion we ignored the cavity decay as this is small on the time scale of the interaction time between a single atom and the cavity field. To observe decoherence we prepare the field in a coherent superposition of coherent states, as previously described, and then let it evolve for a time so that there is a significant probability that at least one photon is lost from the cavity. We then need to find a way to probe the field state at the end of that time. It was not possible to directly measure the state of a microwave cavity field directly. Instead the Haroche experiment used a second atom as a probe for the field state.

There are two possible results for the first atomic measurement (g, e) with the corresponding conditional states given by (15.57 and 15.58). Now consider sending in another atom after some delay time T and ask (for example) for the probability to find the second atom in the excited state given one or the other of the two conditional states, that is to say, we seek the conditional probabilities $p(e|g)$ and $p(e|e)$ (where the conditioning label refers to the result of the first atom measurement). As the respective conditional states are different, these probabilities should be different. The extent of the difference is given by

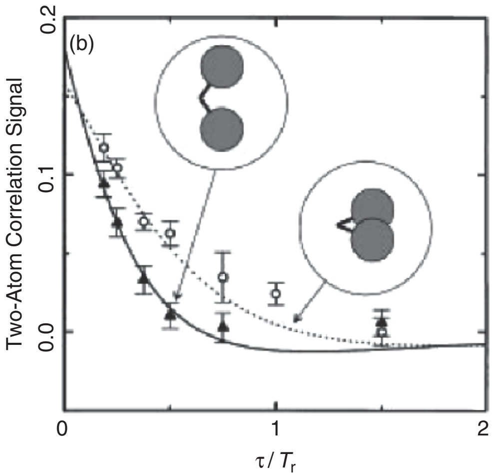

Fig. 15.5 A plot of the two-atom correlation η versus the delay time between successive atoms for two different values of the conditional phase shift. The theoretical results are shown as a dashed and solid line. From Haroche et al. Phys. Rev. Letts., **77**, 4887, (1996), Fig. 4b

$$\eta = p(e|e) - p(e|g) \tag{15.59}$$

If the second atom is sent in a time T after the first the two conditional states, including decoherence, after this delay time can be written in the general form

$$|\psi^{g}_{e}\rangle = \mathcal{N}_{\pm}^{2}\left(|\alpha(t)e^{i\phi}\rangle\langle\alpha(t)e^{i\phi}| + |\alpha(t)e^{-i\phi}\rangle\langle\alpha(t)e^{-i\phi}|\right)$$
$$\pm\mathscr{D}\left(|\alpha(t)e^{i\phi}\rangle\langle\alpha(t)e^{-i\phi}| + |\alpha(t)e^{-i\phi}\rangle\langle\alpha(t)e^{i\phi}|\right)$$

where $\mathscr{D}$ is a measure of the decoherence. In the limit that $\mathscr{D} \to 0$, these two states are indistinguishable and the two conditional probabilities $p(e|g)$ and $p(e|e)$ are the same, so that $\eta \to 0$. Thus by repeating a sequence of double atom experiments the relevant conditional probabilities may be sampled and a value of η as a function of the delay time can be determined.

In Fig. 15.5 we reproduce the results of the experimental determination of η for two different values of the conditional phase shift, ϕ, as a function of the delay time in units of cavity lifetime. As expected the correlation signal decays to zero. Furthermore it decays to zero more rapidly for larger conditional phase shifts, that is to say it decays to zero more rapidly when the superposed states are further apart in phase-space. The agreement with the theoretical result is very good.

15.3 Quantum Measurement Theory

The object of any physical theory is to provide an explanation for the results of measurements. It is usually the case that measurements are made by coupling a macroscopic device to the system of interest which may be of any size, see Fig. 15.6. If the system is very small then some element of amplification is required. Can this process, considered purely as a physical interaction between systems, be described

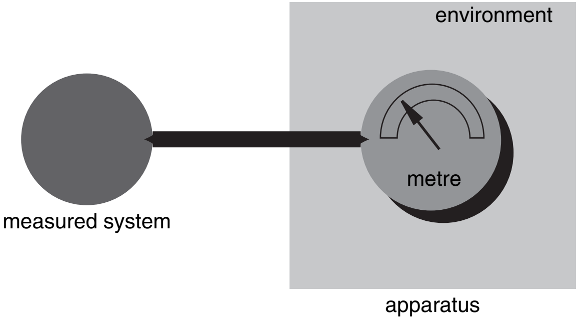

Fig. 15.6 A schematic illustration of a generic measurement illustrating that a measured system is an open system. The metre is a subsystem of the entire measurement apparatus which includes many other degrees of freedom labeled "environment". A particular metre variable, the "pointer" position is observed and will fluctuate in time around an average pointer position

entirely within the framework of quantum mechanics? Would such a description, if given, accord with our intuitive understanding of real measuring devices? How is the measured system effected by the measurement? These are questions which are generally included under the heading of *the measurement problem*.

We will primarily be concerned with measurements that are continuous in time, that is to say, a measurement for which the measurement results are a stochastic process. An example of such a measurement in quantum optics is balanced homodyne detection for which the measurement record is the difference photocurrent. For any measurement we can ask two generic questions:

1. What is the measured system state *averaged* over all measured results: *unconditional dynamics*.
2. What is the measured system state *given* a record of measured results: *conditional dynamics*.

We will take the first question to begin with.

15.3.1 General Measurement Theory

The unconditional state after the measurement is related to the initial state of the system via a *completely positive map* (CP map) which takes density operators to density operators.

$$\rho_0 \to \rho' = \Phi[\rho_0] \tag{15.60}$$

The Krauss representation theorem [2] enables us to write all CP maps as

$$\Phi[\rho_0] = \sum_\alpha \hat{E}_\alpha \rho \hat{E}_\alpha^\dagger \tag{15.61}$$

where

$$\sum_{\alpha} \hat{E}_{\alpha}^{\dagger} \hat{E}_{\alpha} = 1 \tag{15.62}$$

ensures normalisation. The sum in (15.61) could stand for an integral. The Krauss decomposition is not unique: there is a unitary equivalence class of possible Krauss decompositions.

Suppose we measure some metre quantity for which the result is a real number x. The statistics of measurement results is given in terms of a positive operator, $\hat{F}(x)$ as

$$P(x) = \mathrm{tr}(\rho \hat{F}(x)) \tag{15.63}$$

which is the most general way to construct probability distributions from density operators allowed by the quantum formalism. The conditional state *given* the measurement result is,

$$\rho^{(x)} = \frac{\hat{\Upsilon}(x)\rho\hat{\Upsilon}^{\dagger}(x)}{P(x)} \tag{15.64}$$

where $\hat{F}(x) = \hat{\Upsilon}^{\dagger}(x)\hat{\Upsilon}(x)$. The *unconditional* state is given by,

$$\rho' = \int \mathrm{d}x P(x)\rho^{(x)} = \int \mathrm{d}x \hat{\Upsilon}(x)\rho\hat{\Upsilon}^{\dagger}(x) \tag{15.65}$$

which relates the CP map to the measurement operators.

As an example of this formalism we consider the measurement of a generic two level system (a qubit) coupled to a measurement apparatus for which the pointer variable has real eigenvalues. To specify the nature of the measurement we need only specify the Krauss measurement operator $\hat{\Upsilon}(x)$,

$$\hat{\Upsilon}_{\Delta}(x) = (2\pi\Delta)^{-1/4} \exp\left[-\frac{(x-\kappa\hat{\sigma}_z)^2}{4\Delta}\right] \tag{15.66}$$

where $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$ is the measured system observable and $\{|0\rangle, |1\rangle\}$ are an orthonormal basis for the two level system. (see Exercise 15.5). The probability density for measurement result, $x(t)$ at time t;

$$P(x,t) = \mathrm{tr}(\rho(t)\hat{\Upsilon}_{\Delta}^{\dagger}(x)\hat{\Upsilon}_{\Delta}(x)) \tag{15.67}$$

The moments of this distribution are related to the quantum moments of the measured system observable by

$$\begin{aligned} \mathscr{E}(x(t)) &= \kappa\langle\hat{\sigma}_z(t)\rangle \\ \mathscr{E}((\Delta x(t)))^2 &= \Delta + \kappa^2\langle(\Delta\hat{\sigma}_z(t))^2\rangle \end{aligned}$$

Thus we regard Δ as the noise added to the signal by the apparatus. Using the fact that $\sigma_z^2 = 1$ the measurement operator may be written

$$\hat{\Upsilon}_{\Delta}(x) = \sqrt{P_1(x)}|1\rangle\langle 1| + \sqrt{P_0(x)}|0\rangle\langle 0| \tag{15.68}$$

where

$$P_\alpha(x) = (2\pi\Delta)^{-1/2} \exp\left[-\frac{(x+(-1)^\alpha\kappa)^2}{2\Delta}\right] \tag{15.69}$$

where $\alpha = 1, 0$. The unconditional state, is then given by (15.65) as

$$\rho' = p_1|1\rangle\langle 1| + p_0|0\rangle\langle 0| + \mathrm{e}^{-\kappa^2/2\Delta}(|1\rangle\langle 0| + |0\rangle\langle 1|) \tag{15.70}$$

where $p_\alpha = \langle\alpha|\rho|\alpha\rangle$, and $\alpha = 0, 1$. We can now define the *good measurement* limit as $\mathrm{e}^{-\kappa^2/2\Delta} << 1$, and the measurement result statistics easily enable the eigenstates of σ_z to be resolved. In this limit the unconditional post measurement state is diagonalised in the eigenbasis of σ_z.

15.3.2 The Pointer Basis

How does the construction of a physical measurement apparatus determine what system quantity is measured? This is one of the key components of the measurement problem. To understand why there might be a problem we will consider a simple model of the interaction between a measured system S and a measuring device M. Let us first ignore the macroscopic nature of M and simply treat it as a single quantum system with one degree of freedom. As we shall see such an assumption does not lead to an adequate description of the measurement process. The macroscopic nature of M is essential for a complete description.

Let $\{|m_i>\}$ denote a set of eigenstates of some meter quantity and let $\{|s_i>\}$ denote the eigenstates which diagonalise the system operator we are seeking to measure. Suppose the initial state of the system is $|m_0> \otimes |s_i>$, and further suppose that under unitary evolution this state evolves to

$$|\psi_i(t)> = |m_i> \otimes |s_i> \tag{15.71}$$

Then for a general system state

$$|\psi> = \sum_i c_i |s_i> \otimes |m_0> \tag{15.72}$$

the state at time t will be

$$|\psi(t)> = \sum_i c_i |m_i> \otimes |s_i> \tag{15.73}$$

We regard m_i as the value read-out from the meter scale. If an observer finds the meter in a state $|m_j>$ (with probability $|c_j|^2$) the system is subsequently (i.e. conditionally) described by the state $|s_j>$. It is clear that in effect we have measured some physical quantity S which is diagonal in the basis $\{|s_i>\}$,

$$S = \sum_i a_i |s_i >< s_i| \tag{15.74}$$

Suppose one only knows that a measurement has taken place but we do not select a particular result. The state of the system must then be described by the reduced density operator obtained by tracing out over the meter states,

$$\rho_s = \sum_i |c_i|^2 |s_i >< s_i| \tag{15.75}$$

Thus as a result of the measurement the density operator of the system is diagonal in the basis which diagonalises the measured operator. Alternatively the basis in which ρ_s is diagonal determines the system operator which has been measured. This is the standard description of the measurement process. Unfortunately it is inadequate as we now explain.

The description given above assumed we read out a meter quantity which is diagonal in the basis $\{|m_i >\}$. Suppose however we decide to read out another meter quantity diagonal in the basis $\{|\tilde{m}_i >\}$. We can then express the meter states $|m_i >$ in the alternative basis $\{|\tilde{m}_i >\}$,

$$|m_i >= \sum_j < \tilde{m}_j | m_i > |\tilde{m}_j > \tag{15.76}$$

The state of the combined system after the interaction is then

$$\begin{aligned} |\psi(t) > &= \sum_i c_i |m_i > \otimes |s_i > \\ &= \sum_j d_j |\tilde{m}_j > \otimes |r_j > \end{aligned} \tag{15.77}$$

where

$$d_j |r_j >= \sum_i c_i < \tilde{m}_j | m_i > |s_i > \tag{15.78}$$

Although the system meter coupling has not altered there is now some ambiguity as to what system operator has been measured; is it

$$S = \sum_i a_i |s_i >< s_i|$$

or

$$R = \sum_i b_i |r_i >< r_i| \quad ? \tag{15.79}$$

This ambiguity can only be removed if we say that the meter is so constructed that the only physical property that we readout is the one which is diagonal in the basis $\{|m_i >\}$. What is the property of the meter which determines such a preferred or *pointer* basis?

We can answer this question by admitting that a true measuring device must be macroscopic and thus contain many degrees of freedom. Thus, in addition to the read-out variable, we must consider the meter as composed of many other degrees of freedom possibly coupled to the the read-out variable. These other degrees of freedom cannot be determined in any realistic scheme and thus may be treated as a environment. In order to model this situation we now divide the measurement scheme into system+meter+environment. The system and meter are directly coupled and the meter is coupled to the environment, see Fig. 15.6. As Zurek has shown it is the nature of the coupling between the meter and the reservoir which determines the pointer basis.

In order that the special correlation between the system and the meter given by (15.73) be preserved in the presence of the coupling to the reservoir, described by some Hamiltonian H_{ME}, we require that the pointer basis $\{|m_i>\}$ be a complete set of eigenstates for a pointer quantity M which commutes with H_{ME}. This ensures that fluctuations from the reservoir do not find there way back to the measured system quantity S. In many situations it may not be possible to find an operator which satisfies this condition exactly. However an approximate pointer basis may exist in as much as the diagonal elements of the density operator in such a basis relax on a very long time scale while the off-diagonal elements decay on a much shorter time scale.

Thus it is the meter-reservoir interaction which determines the pointer observable M and thus the corresponding pointer basis appropriate to the measurement. The to-be-measured quantity is only defined in the course of the meter-reservoir interaction; a situation consistent with Bohr's general description of measurement in quantum mechanics.

The meter cannot be observed in a superposition of pointer basis states as its state vector is being continually collapsed. It is the monitoring of the meter by the reservoir which results in the apparent state reduction of the system and the meter is so constructed to ensure that this occurs. The correlations between the corresponding system and pointer basis states are preserved in the final mixed state density operator

$$\begin{aligned}&\sum_{i,j} c_i c_j^* |m_i><m_j| \otimes |s_i><s_j| \rightarrow \\ &\quad \sum_i |b_i|^2 |\tilde{m}_i><\tilde{m}_i| \otimes |r_i^p><r_i^p| \end{aligned} \tag{15.80}$$

where $\{|r_i^p>\}$ are the system states determined by the pointer basis, referred to as the *relative states*.

There is a close connection between the concept of a quantum nondemolition measurement and the concept of a pointer basis. The condition that an operator $Q(t)$ be a QND variable is that (in the interaction picture)

$$[Q(t), Q(t')] = 0 \tag{15.81}$$

Applying this idea to the measurement description above it is clear that the pointer observable P, which determines the pointer basis, must be a QND variable of the meter. This ensures that an initial eigenstate of P evolves entirely within the pointer basis set.

In the theory of QND measurements we also require the variable Q to maintain its QND property in the presence of couplings to other systems, the reservoir in our example, which represent further stages of the meter device. This will be true provided the back action evasion criterion is satisfied

$$[Q(t), H_{\mathrm{ME}}] = 0 \tag{15.82}$$

where H_{ME} represents the interaction Hamiltonian between the system of the QND variable and other systems to which it is coupled (the reservoir in our case). This property is precisely the property that the pointer observable must satisfy. We may thus view the pointer observable as a QND variable of the meter which is coupled to the environment by a back action evasion coupling.

15.4 Examples of Pointer Observables

In Sect. 15.1 we considered a number of models which lead to the density operator becoming diagonal in a preferred basis. For example, if the meter is a harmonic oscillator coupled to the bath by

$$H_{\mathrm{ME}} = a^\dagger a \Gamma_E \tag{15.83}$$

so that it evolves according to the master equation (15.46) the state of the meter becomes diagonal in the number basis. The pointer observable is $a^\dagger a$ and (15.83) represents an ideal back action evasion coupling.

As another example suppose that the amplitude of an oscillator is coupled to the environment by

$$H_{\mathrm{ME}} = a\Gamma^\dagger + a^\dagger \Gamma \tag{15.84}$$

This is not a back action evasion coupling, however the meter state tends to become diagonal in the coherent state basis which are eigenstates of the operator a. Unfortunately the diagonal elements of the density operator in this basis are also changed. However as we saw in (15.43) for short times the diagonal elements do not change much, yet coherence between states separated by large coherent amplitudes decay quite rapidly. In this case we can say that the coherent states are an approximate pointer basis.

15.5 Model of a Measurement

We are now able to consider a full model of a quantum limited measurement including the interaction of the meter with the environment. The quantum system and meter are taken to be harmonic oscillators with annihilation operators a and b

respectively. The coupling between the system and the meter is taken to be quadratic in the system amplitude and the interaction Hamiltonian is

$$H_{\mathrm{SM}} = \frac{\hbar}{2} a^\dagger a (bE^* + b^\dagger E) \; . \tag{15.85}$$

Such a system may represent a four wave mixing interaction in quantum optics, where one field is taken as classical of amplitude E. The measured system operator is, as we shall see, $a^\dagger a$ (or some function thereof).

We assume that mode b is coupled to the environment by amplitude coupling

$$H_{ME} = b\Gamma^\dagger + b^\dagger \Gamma \; . \tag{15.86}$$

This will determine a particular pointer basis. There are good physical reasons why this is a suitable choice for H_{ME}. First, if the oscillators are realised as field modes this coupling represents the usual system-bath interaction of a linear loss mechanism. In particular it could represent the interaction of a field mode with a photoelectron counter. Perhaps the most important reason for choosing H_{ME} in this form however is that it leads to the coherent state pointer basis. As coherent states have a well defined semiclassical limit this is a desirable basis for a real (i.e. classical) measuring device. We now solve for the complete dynamics of the system and meter coupled to the bath.

The density operator for the system and meter after tracing out the reservoir obeys the master equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{-\mathrm{i}}{2}[(Eb^\dagger + E^* b)a^\dagger a, \rho] + \frac{\gamma}{2}(2b\rho b^\dagger - b^\dagger b\rho - \rho b^\dagger b) \tag{15.87}$$

We have assumed the environment is at zero temperature. Initially the state of the system is arbitrary while the meter is in the ground state

$$\rho(0) = \sum_{n,m=0}^{\infty} (\rho_{nm}|n><m|)_S \otimes (|0><0|)_M \tag{15.88}$$

where we have expanded the system state in energy eigenstates and $\rho_{nm} =< n|\rho_s(0)|m >$. Equation (15.87) may be solved by the characteristic function. The solution is,

$$\rho(t) = \sum_{n,m=0}^{\infty} \exp\left(\frac{|E|^2}{\gamma^2}(n-m)^2(1-\gamma t/2 - \mathrm{e}^{-\gamma t/2}) \right) \times (|n><m|)_S \otimes \left(\frac{|\alpha_n(t)><\alpha_m(t)|}{<\alpha_m(t)|\alpha_n(t)>} \right)_M \tag{15.89}$$

where $|\alpha_n(t)>$ is a meter coherent state with

$$\alpha_n(t) = \frac{E_n}{\gamma}(1-\mathrm{e}^{-\gamma t/2}) \tag{15.90}$$

In the long time limit we have

$$\rho \to \sum_n P(n)\left(|n><n|\right)_S \otimes \left(|\alpha_n><\alpha_n|\right)_M \tag{15.91}$$

where $P(n)$ is the initial number distribution for the system. This is a mixture of number states in the system, perfectly correlated with a mixture of coherent states in the meter. It is thus of the general form of (15.80). The coherent states $|\alpha_n>$ are the pointer basis states and the number states the corresponding relative states. The amplitude of the coherent states can be made arbitrarily large by increasing the strength of the system meter coupling E. Hence this model includes amplification in a natural way. The large E limit is the appropriate limit for an accurate measurement [3]. In fact the states $|\alpha_n>$ for different values of n become approximately orthogonal as the coupling strength is increased.

We have assumed in this analysis that the environment is at zero temperature. For photoelectric detection this is a good assumption at optical frequencies. Were the environment not taken at zero temperature there would be an additional thermal spread in the diagonal part of the meter states.

This model contains all the features of the measurement process discussed in Sect. 15.3.2. The correlations between the system and the meter are created by unitary evoultion. The (almost complete) reduction of the meter states to the pointer basis (the coherent states) occurs as a result of nonunitary dissipative evolution, which causes the off-diagonal elements of the meter state in the pointer basis to decay. It is clear that the model represents the measurement of some function of the system number operator $a^\dagger a$. To determine what this function is we must reconsider the interpretation of the environment as a photoelectron counter. If we assume that every meter quanta lost to the environment is actually counted in the detector, a full analysis shows that in fact the model describes the measurement of the square of the number operator.

From (15.91) we can calculate the reduced state of the system by tracing out over the meter states. The resulting state has an exponential decay of off-diagonal elements in the number basis which goes as t^2 for short times. Such a dependence indicates that for short times a Markovian evolution equation does not describe the evolution of the system state. However if the rate of photon counting, γ, is very large then this short time behaviour is rapidly superceeded by an exponential linear dependence. In this case the effective master equation for the system state is

$$\frac{\mathrm{d}\rho_S}{\mathrm{d}t} = -\frac{|E|^2}{2\gamma}\left[a^\dagger a, \left[a^\dagger a, \rho_S\right]\right] \tag{15.92}$$

In this strong measurement limit we see that as far as the system is concerned the effect of the measurement of photon number is to induce a diffusion in the oscillator phase. In some sense the phase is the conjugate variable to the photon number so this result is consistent with the uncertainty principle.

15.6 Conditional States and Quantum Trajectories

We turn now to discuss the second question that may be asked of a measured system: what is the conditional state conditioned on a measurement result? In Chap. 6 we discussed an approach to solving master equations based on unravelling the solution as an average over a quantum trajectory: a solution to a stochastic master equation or stochastic Schrödinger equation. Here we will show that a quantum trajectory can describe the conditional evolution of the state of a continuously measured system, conditioned on the stochastic measurement record.

A simple model for continuous can be built out of the two-level mode discussed in Sect. 15.3.1. To get to a continuous measurement we let readouts occur at Poisson distributed times at rate γ. The system then obeys the master equation

$$\begin{aligned}\dot{\rho} &= -\mathrm{i}[H,\rho] + \gamma\left(\int_{-\infty}^{\infty} \mathrm{d}x \hat{\Upsilon}_\Delta(x)\rho\hat{\Upsilon}_\Delta^\dagger(x) - \rho\right) \\ &= -\mathrm{i}[H,\rho] - \frac{\gamma}{4}\left(1 - \mathrm{e}^{-\kappa^2/2\Delta}\right)[\hat{\sigma}_z,[\hat{\sigma}_z,\rho]]\end{aligned}$$

Clearly this describes a QND measurement of σ_z. We now take the limit of weak, rapid measurements $\gamma >> 1,\ \ \Delta >> \kappa^2$ to get

$$\dot{\rho} = -\mathrm{i}[H,\rho] - D[\hat{\sigma}_z,[\hat{\sigma}_z,\rho]] \tag{15.93}$$

where $D = \frac{\gamma\kappa^2}{8\Delta}$ is the *decoherence* rate in the $\hat{\sigma}_z$ basis. When D is large decoherence is rapid, which should correspond to a good measurement.

The measurement record is a real valued classical stochastic variable, $x(t_i)$, conditioned on the state of the system. To get a continuous stochastic record we define a stochastic differential: $\mathrm{d}y(t) = \mathrm{d}N(t)x(t)$ where $\mathrm{d}N(t)$ is a Poisson process:

$$\begin{aligned}\mathrm{d}N(t)^2 &= \mathrm{d}N(t) \\ \mathscr{E}(\mathrm{d}N(t)) &= \gamma\mathrm{d}t\end{aligned}$$

We can now take the *diffusive limit.* Consider a time δt such that $\gamma\delta t >> 1$ yet, $D\delta t << 1$.

$$\mathscr{E}(y(t+\delta t) - y(t)) \approx \gamma\delta t\kappa\langle\hat{\sigma}_z(t)\rangle_c \tag{15.94}$$

$$\mathscr{E}((y(t+\delta t) - y(t))^2 \approx \gamma\delta t\Delta \tag{15.95}$$

we can then approximate the observed process by the Ito stochastic d.e.

$$\mathrm{d}y(t) = \gamma\kappa\left[\langle\hat{\sigma}_z(t)\rangle_c\mathrm{d}t + (8D)^{-1/2}\mathrm{d}W(t)\right] \tag{15.96}$$

where $\mathrm{d}W(t)$ is the Wiener process: $\mathscr{E}(\mathrm{d}W(t)) = 0,\ \ \mathscr{E}(\mathrm{d}W(t)^2) = \mathrm{d}t$. If D is large, then we will have a good signal-to-noise ratio.

Under the same assumptions we can derive the conditional master equation and the stochastic Schrödinger equation for this measurement. Using (15.64 and 15.67) and taking the limit of weak rapid measurement as above we find that the conditional master equation is

$$d\rho_c(t) = -i[H,\rho_c(t)]dt - Ddt[\sigma_z,[\sigma_z,\rho_c(t)]] + \sqrt{2D}dW(t)\mathscr{H}[\sigma_z]\rho_c(t) \quad (15.97)$$

and the stochastic Schrödinger equation is

$$d|\psi_c(t)\rangle = -iHdt|\psi_c(t)\rangle - Ddt\left[1 - 2\langle\sigma_z\rangle_c\sigma_z + \langle\sigma_z\rangle^2\right]|\psi_c(t)\rangle + \sqrt{2D}(\sigma_z - \langle\sigma_z\rangle_c)dW(t)|\psi_c(t)\rangle \quad (15.98)$$

This leads us to define a *quantum limited measurement* as measurement for which the signal-to-noise is determined only by the intrinsic quantum noise of the measured system, and the minimum back-action noise consistent with the uncertainty principle. Under these conditions the decoherence rate is determined *only* by the back-action noise due to measurement. The example considered here is an example of a quantum limited measurement. Usually a measurement is not quantum-limited as the system dynamics has other irreversible channels (e.g. dissipation) and other sources of noise (e.g. thermal) are added to the measured signal.

The stochastic master equation has an interesting property. Using (15.97) we can derive equations of motion for the conditional mean value of σ_z. Define $z_c = \langle\sigma_z\rangle_c$

$$\mathrm{d}z_c = (\ldots)\mathrm{d}t + 2\sqrt{D}(1 - z_c^2)\mathrm{d}W(t) \quad (15.99)$$

where $\ldots$ refers to that part of the dynamics arising directly from the Hamiltonian term. If we assume that $[\sigma_z, H] = 0$, the stochastic dynamics is determined entirely by the measurement. It is then clear that there are two fixed points, $z_c = \pm 1$ which correspond to two eigenstates of σ_z. In a simulation the result is that the conditional state tends to localise on one or the other of the eigenstates of σ_z. This is the continuous measurement version of quantum state reduction.

15.6.1 Homodyne Measurement of a Cavity Field

Consider now a single mode cavity damped into a zero temperature heat bath (see Sect. 6.1). The environment is the multi-mode field external to the cavity. Some of these external modes are coupled into a balanced homodyne detection system. The change in the state of the system, over a time interval $\mathrm{d}t$, can be described by a single jump operator a. In what follows we choose the units of time so that the cavity decay constant, γ is unity. We can define two Krauss operators. Firstly we define $\Omega_1 = a\sqrt{\mathrm{d}t}$, corresponding to a conditional Poisson process with probability $\langle a^\dagger a\rangle\mathrm{d}t$. Normalization requires a second Krauss operator, $\Omega_0 = I - a^\dagger a\mathrm{d}t/2 - \mathrm{i}H\mathrm{d}t$, where H is Hermitian. Then the unconditional master equation without feedback is just the familiar Lindblad form

$$\begin{aligned} d\rho &= \Omega_0\rho\Omega_0 + \Omega_1\rho\Omega_1 - \rho \\ &= -\mathrm{i}[H,\rho]dt + a\rho a^\dagger dt - \frac{1}{2}(a^\dagger a\rho + \rho a^\dagger a)dt \\ &\equiv -\mathrm{i}[H,\rho]dt + \mathscr{D}[a]\rho dt\ . \end{aligned} \tag{15.100}$$

A fermionic example is given in [4].

In homodyne detection, the output field from the cavity is mixed with a strong coherent field, the local oscillator. This corresponds to a displacement of the cavity field a. Obviously this does not change the unconditional dynamics of the cavity field. With this in mind we first note that given some complex number $\alpha = |\alpha|e^{i\phi}$, we may make the transformation

$$\begin{aligned} a &\to a + \alpha \\ H &\to H - \frac{\mathrm{i}|\alpha|}{2}(e^{-\mathrm{i}\phi}a - e^{i\phi}a^\dagger) \end{aligned} \tag{15.101}$$

and obtain the same master equation. In the limit as $|\alpha|$ becomes very large, the rate of the Poisson process is dominated by the term $|\alpha|^2$. In this case it may become impossible to monitor every jump process, and a better strategy is to approximate the Poisson stochastic process by a Gaussian white-noise process.

For large α, we can consider the system for a time δt in which the system changes negligibly but the number of detections $\delta N(t) \approx |\alpha|^2\delta t$ is very large; then we can approximate $\delta N(t)$ as

$$\delta N(t) \approx |\alpha|^2\delta t + |\alpha|\langle e^{-i\phi}a + a^\dagger e^{i\phi}\rangle_c\delta t + |\alpha|\delta W(t), \tag{15.102}$$

where $\delta W(t)$ is normally distributed with mean zero and variance δt (a *Wiener increment*).

We now define the stochastic measurement record as the current

$$I_c^{\mathrm{hom}}(t) = \lim_{\alpha\to\infty} \frac{\delta N(t) - |\alpha|^2\delta t}{|\alpha|\delta t} \tag{15.103}$$

$$= \langle e^{-\mathrm{i}\phi}a + e^{\mathrm{i}\phi}a^\dagger\rangle + dW(t)/dt\ . \tag{15.104}$$

In balanced homodyne detection, this current corresponds to the photo-current. Given this stochastic measurement record, we can determine the conditional state of the quantum system by the stochastic master equation (SME)

$$\begin{aligned} d\rho_a(t) = &-\mathrm{i}\,[H, \rho_c(t)]\,dt + \mathscr{D}[e^{-\mathrm{i}\phi}a]\rho_c(t)dt \\ &+ \mathscr{H}[e^{-\mathrm{i}\phi}a]\rho_c(t)dW(t). \end{aligned} \tag{15.105}$$

In the above equations, the expectation $\langle a\rangle_c$ denotes the average over the conditional state, $\mathrm{Tr}(\rho_c a)$ and $\mathscr{H}$ is a superoperator defined in Sect. 6.7

$$\mathscr{H}[c]\rho = c\rho + \rho c^\dagger - \rho\,\mathrm{Tr}[c\rho + \rho c^\dagger]\,. \tag{15.106}$$

Let us define the quadrature phase operator

$$X_\theta = \mathrm{e}^{-\mathrm{i}\phi} a + \mathrm{e}^{\mathrm{i}\phi} a^\dagger \tag{15.107}$$

The corresponding stochastic Schrödinger equation is

$$\begin{aligned} \mathrm{d}|\psi_c(t)\rangle = &\Big\{ -\mathrm{i}H\mathrm{d}t - \frac{1}{2}\left[a^\dagger a - 2\langle X_\theta/2\rangle_c + \langle X_\theta/2\rangle_c^2\right]\mathrm{d}t \\ &+ \left[a - \langle X/2\rangle_c\right]\mathrm{d}W(t)\Big\}\,|\psi_c(t)\rangle \end{aligned} \tag{15.108}$$

Note that as $|\psi_c(t)\rangle$ approaches a coherent state for which $a|\psi_c(t)\rangle = \langle X/2\rangle_c|\psi_c(t)\rangle$ the noise term tends to zero. This leads to the stochastic localisation of the conditional state on the set of coherent states. If we ignore normalisation of the state, we get a very direct sense of how the measured photo current conditions the state,

$$\mathrm{d}|\bar{\psi}_c(t)\rangle = \mathrm{d}t\left[-\mathrm{i}H - \frac{1}{2}a^\dagger a + I_c^{\mathrm{hom}}(t)\right]|\bar{\psi}_c(t)\rangle \tag{15.109}$$

Exercises

15.1 Consider the superposition of two coherent states

$$|\psi\rangle = \mathscr{N}(|\alpha_0\rangle + |-\alpha_0\rangle)$$

where α_0 is real. Show that the probability distribution for $X_1 = a + a^\dagger$ is a double peaked Gaussian ($\alpha_0 > 1$) while the distribution for $X_2 = -\mathrm{i}(a - a^\dagger)$ shows interference fringes.

15.2 Consider the Hamiltonian

$$H = \hbar\omega a^\dagger a + \hbar\chi(a^\dagger a)^2 \tag{15.110}$$

Show that this Hamiltonian can generate a superposition of two coherent states or two squeezed states of the type discussed in the preceeding questions.

15.3 The spin coherent states are defined by

$$|j;\gamma\rangle = \exp\left(-\frac{\theta}{2}(J_+\mathrm{e}^{-\mathrm{i}\phi} - J_-\mathrm{e}^{\mathrm{i}\phi})\right)|j,j\rangle \tag{15.111}$$

where $\gamma = \mathrm{e}^{\mathrm{i}\phi}\tan\frac{\theta}{2}$ and $|j,j\rangle$ is a J_z eigenstate with eigenvalue j . Consider the superposition state

$$|\psi\rangle = \mathscr{N}(|j;\gamma_0\rangle + |j;-\gamma_0\rangle) \tag{15.112}$$

with γ_0 real. Calculate the probability distributions for J_x, J_y, J_z and show that the J_y distribution exhibits interference effects. Give a geometrical interpretation of this result.

15.4 Consider the two mode squeezed vacuum state

$$|\psi>= (\cosh\ r)^{-1} \sum_{n=0}^{\infty} (\tanh\ r)^n\ |n>_1 \otimes |n>_2 \tag{15.113}$$

and the mixed state with the same diagonal distribution

$$\rho = (\cosh\ r)^{-2} \sum_{n=0}^{\infty} (\tanh\ r)^{2n} |n>_1< n| \otimes |n>_2< n| \tag{15.114}$$

Show that these states may be distinguished by the probability distributions for the two mode quadrature phase operators $X_\pm = a_\pm + a_\pm^\dagger$ where

$$a_\pm = \frac{1}{\sqrt{2}}(a_1 \pm a_2)\ . \tag{15.115}$$

15.5 Consider a two-level system with basis states $\{|0\rangle, |1\rangle\}$ coupled impulsively to the momentum of a free particle via the Hamiltonian

$$H_I(t) = \kappa \hat{p} \sigma_z \delta(t - t_r) \tag{15.116}$$

with $\sigma_z = |1\rangle\langle 1| - |0\rangle\langle 0|$. Just prior to the readout at time t_r the state of the particle is a Gaussian with the wave function $\psi(p) = (2\pi\Delta)^{-1/4} \exp[-x^2/(4\Delta)]$. Immediately after the coupling at time t_r the position of the particle is measured with perfect accuracy and projected onto the position eigenstate $|x_r\rangle$. Show that the conditional state of the two level system immediately after the readout is given by (15.70) with $x = x_r$.

15.6 Show that the measurement operator

$$\hat{\Upsilon}_\Delta(x) = (2\pi\Delta)^{-1/4} \exp\left[-\frac{(x - \kappa\hat{\sigma}_z)^2}{4\Delta}\right] \tag{15.117}$$

reduces to a projection operator in the limit $\sigma \to 0$.

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